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FLEXURAL VIBRATIONS OF CYLINDRICALLY  
AEOLOTROPIC CIRCULAR PLATES

by

I. A. Minkarah

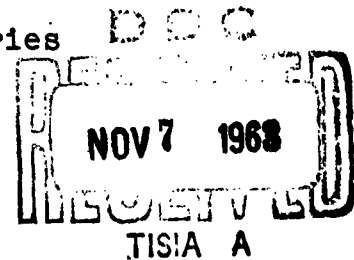
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ABSTRACT

Equations for the determination of the frequencies of flexural vibration and corresponding modal shapes for cylindrically aeolotropic circular plates are presented. A considerable analysis has been made of the case of symmetric vibrations and results are tabulated.

The theory is exemplified by an analysis of the vibration of a circular plate with circular stiffeners for which the effective elastic compliances had been determined in a previous investigation. Nodal patterns and corresponding frequencies were experimentally determined for the first five symmetric modes and the first four asymmetric modes of a simply supported stiffened plate. Comparison between the theoretical and experimental results for the first three symmetric modes is provided.

Finally, some discussion is given for the important problem of approximating plates with attached stiffeners by equivalent homogeneous uniformly thick aeolotropic plates.

## INTRODUCTION

The literature contains many analyses of transverse elastic vibrations of plates from the standpoint of small deflection, thin-plate theory<sup>1</sup>. Past research deals with both isotropic and orthotropic plates but mainly plates which are rectangular in shape. At least one paper<sup>2</sup> in the recent past deals with the vibrations of circular and elliptical orthotropic plates for which the principal axes of orthotropy are parallel or perpendicular to a given diameter in the circular plates and parallel or perpendicular to a principal diameter in the elliptical plates.

In the present paper both a theoretical and experimental study is made of the flexural vibration of circular plates which are cylindrically aeolotropic. That is the principal directions of aeolotropy at a point are in radial and transverse directions. Such aeolotropy may occur in nature as in

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<sup>1</sup>N. J. Huffington, Jr. and W. H. Hoppmann II, "On the Transverse Vibrations of Rectangular Orthotropic Plates", J. App. Mechanics, 25, No. 3, 389-395 (1958).

<sup>2</sup>W. H. Hoppmann II, "Flexural Vibrations of Orthogonally Stiffened Circular and Elliptical Plates", Proc. of 3rd U.S. Nat. Cong. of App. Mechanics, 181-187 (1958).

some cuts of wood or may be manufactured at least approximately in the case of plates of reinforced materials or plates with stiffeners attached.

# DIFFERENTIAL EQUATION AND ITS SOLUTION FOR THE FREE FLEXURAL VIBRATION OF CYLINDRICALLY AEOLOTROPIC PLATES

The differential equation for the transverse statical deflection of cylindrically aeolotropic plates is in the literature<sup>3,4,5</sup>. Its extension to the case of transverse flexural vibration is, as usual, obtained simply by adding the inertia term. It may be written as follows:

$$D_r \frac{\partial^4 w}{\partial r^4} + \frac{2D_{r\theta}}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{D_\theta}{r^4} \frac{\partial^4 w}{\partial \theta^4} + \frac{2D_r}{r} \frac{\partial^3 w}{\partial r^3} - \frac{2D_{r\theta}}{r^3} \frac{\partial^3 w}{\partial r \partial \theta^2} -$$

$$- \frac{D_\theta}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{2}{r^4} (D_\theta + D_{r\theta}) \frac{\partial^2 w}{\partial \theta^2} + \frac{D_\theta}{r^3} \frac{\partial w}{\partial r} + \frac{\gamma h}{g} \frac{\partial^2 w}{\partial t^2} = 0 \quad (1)$$

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<sup>3</sup>G. F. Carrier, "The Bending of the Cylindrically Aeolotropic Plate", J. App. Mechanics, vol. 11, No. 3, 129 (1944).

<sup>4</sup>A. M. Sen Gupta, "Bending of a Cylindrically Aeolotropic Circular Plate with Eccentric Load", J. App. Mechanics, vol. 19, No. 1, 9 (1952).

<sup>5</sup>S. G. Lekhnitski, "Anisotropic Plates", translated by E. Z. Stowell, American Iron and Steel Institute, New York, June 1956. Contribution to the Metallurgy of Steel, No. 50, p.22.

Equation (1) may be solved by the method of separation of space and time variables. Let  $w = \phi(r, \theta) e^{i\omega t}$  and  $c_1^2 = \frac{g}{\gamma h}$  then Eq. (1) becomes:

$$\begin{aligned} D_r \frac{\partial^4 \phi}{\partial r^4} + \frac{2D_{r\theta}}{r^2} \frac{\partial^4 \phi}{\partial r^2 \partial \theta^2} + \frac{D_\theta}{r^4} \frac{\partial^4 \phi}{\partial \theta^4} + \frac{2D_r}{r} \frac{\partial^3 \phi}{\partial r^3} - \frac{2D_{r\theta}}{r^3} \frac{\partial^3 \phi}{\partial r \partial \theta^2} - \\ - \frac{D_\theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r^4} (D_\theta + D_{r\theta}) \frac{\partial^2 \phi}{\partial \theta^2} + \frac{D_\theta}{r^3} \frac{\partial \phi}{\partial r} - \frac{\omega^2}{c_1^2} \phi = 0 \end{aligned} \quad (2)$$

where  $\omega$  is the circular frequency.

Furthermore we may likewise separate the space variables as follows:

$$\phi = \sum_{v=-\infty}^{\infty} Y(r) e^{iv\theta}$$

Equation (2) then becomes an ordinary differential equation on  $Y(r)$  as follows:

$$\begin{aligned} [D_r \frac{d^4}{dr^4} - \frac{2v^2 D_{r\theta}}{r^2} \frac{d^2}{dr^2} + \frac{v^4 D_\theta}{r^4} + \frac{2D_r}{r} \frac{d^3}{dr^3} + \frac{2v^2 D_{r\theta}}{r^3} \frac{d}{dr} - \frac{D_\theta}{r^2} \frac{d^2}{dr^2} - \\ - \frac{2v^2}{r^4} (D_\theta + D_{r\theta}) + \frac{D_\theta}{r^3} \frac{d}{dr} - \left(\frac{\omega}{c_1}\right)^2] Y(r) = 0 \end{aligned} \quad (3)$$

or

$$\left\{ D^4 + \frac{2}{r} D^3 - \frac{F}{r^2} D^2 + \frac{F}{r^3} D + \frac{G}{r^4} - \alpha^4 \right\} Y(r) = 0 \quad (4)$$

where

$$\sigma' = \frac{D_{r\theta}}{D_r}, \quad k^2 = \frac{D_\theta}{D_r}, \quad F = 2v^2\sigma_1 + k^2,$$

$$\alpha^4 = \frac{\omega^2}{c_1^2 D_r} = \frac{\omega^2 \gamma h}{g D_r}, \quad \text{and} \quad G = v^4 k^2 - 2v^2(k^2 + \sigma')$$

The latter differential equation can be solved in series form.

Let

$$Y(r) = \sum_{n=0}^{\infty} a_n r^{n+m} \quad (5)$$

where  $m$  is the undetermined index of the series.

Then Eq. (4) gives the following equation:

$$\begin{aligned} & [m(m-1)^2(m-2) - m(m-2)F + G]a_0 r^m + [(1+m)(m-1)m^2 - (m+1)(m-1)F + G]a_1 r^{m+1} + \\ & + [(2+m)(1+m)^2 - m(2+m)F + G]a_2 r^{m+2} + \\ & + [(3+m)(m+2)^2(m+1) - (m+1)(3+m)F + G]a_3 r^{m+3} + \\ & + \sum_{n=4}^{\infty} [(n+m)(n+m-1)^2(n+m-2) - F(n+m)(n+m-2) + G]a_n r^{n+m} - \alpha^4 a_{n-4} r^{n+m} = 0 \end{aligned} \quad (6)$$

Therefore, if  $Y(r)$  is to be an integral of Eq. (4), the coefficient of each power of  $r$  in Eq. (6) must be identically zero. Setting the coefficient of  $r^{n+m}$  equal to zero, the

recurrence equation for the coefficients  $a_n$  is:

$$[(n+m)(n+m-1)^2(n+m-2) - F(n+m)(n+m-2) + G]a_n - \alpha^4 a_{n-4} = 0 \quad n \geq 4 \quad (7)$$

which defines all the coefficients in terms of four of them, say  $a_0$ ,  $a_1$ ,  $a_2$ , and  $a_3$  as soon as the index  $m$  is known.

If  $a_0 \neq 0$ , the coefficient of  $r^m$  equated to zero gives the indicial equation:

$$m(m-1)^2(m-2) - m(m-2)F + G = 0 \quad (8)$$

With a change of variable  $m = x + 1$ , Eq. (8) can be reduced to a biquadratic from which the four values for the indices are:

$$m = 1 \pm \sqrt{\frac{(1+F) \pm \sqrt{(1+F)^2 - 4(F+G)}}{2}} \quad (9)$$

Once the elastic compliances of a certain material are known, the four values of  $m$  can be computed from Eq. (9); the recurrence equation for every root can then be obtained, and the series solutions for that particular problem obtained.



# SOLUTIONS FOR THE CASE OF SYMMETRICAL VIBRATIONS

For symmetrical vibrations the differential equation is independent of  $\Theta$  and all terms containing  $\nu$  vanish. The parameters  $F$  and  $G$  become equal to  $k^2$  and  $0$  respectively, and Eq. (4) reduces to:

$$\left[D^4 + \frac{2}{r} D^3 - \frac{k^2}{r^2} D^2 + \frac{k^2}{r^3} D - \alpha^4\right]Y(r) = 0 \quad (10)$$

Substituting  $k^2$  and  $0$  for  $F$  and  $G$  in Eq. (9), the four values of the index become:

$$\begin{aligned} m_1 &= 0 \\ m_2 &= 2 \\ m_3 &= 1 + k \\ m_4 &= 1 - k \end{aligned} \quad (11)$$

The recurrence equation (7) reduces to:

$$(n+m)(n+m-2)[(n+m-1)^2 - k^2]a_n = \alpha^4 a_{n-4} \quad n \geq 4 \quad (12)$$

and

$$a_n = \frac{\alpha^4 a_{n-4}}{(n+m)(n+m-2)[(n+m-1)^2 - k^2]} \quad n \geq 4 \quad (13)$$

The coefficient of  $r^{m+1}$ ,  $r^{m+2}$ ,  $r^{m+3}$  require that  $a_1$ ,  $a_2$  and  $a_3$  be equal to zero. The equation corresponding to  $m = 0$  is:

$$a_n = \frac{\alpha^4 a_0}{n(n-2)[(n-1)^2 - k^2]} \quad n \geq 4$$

and gives:

$$a_4 = \frac{\alpha^4 a_0}{2 \cdot 4(9-k^2)}$$

$$a_5 = a_6 = a_7 = 0$$

$$a_8 = \frac{\alpha^4 a_4}{6 \cdot 8(49-k^2)} = \frac{\alpha^4 \cdot \alpha^4 a_0}{2 \cdot 4 \cdot 6 \cdot 8(9-k^2)(49-k^2)}$$

and

$$\begin{aligned} c_1 Y_1(r) = a_0 & \left[ 1 + \frac{(\alpha r)^4}{2 \cdot 4(9-k^2)} + \frac{(\alpha r)^8}{2 \cdot 4 \cdot 6 \cdot 8(9-k^2)(49-k^2)} + \right. \\ & + \frac{(\alpha r)^{12}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12(9-k^2)(49-k^2)(121-k^2)} + \dots + \\ & \left. + \frac{(\alpha r)^{4j}}{2^{2j} \cdot 2j!(9-k^2)(49-k^2) \dots [(4j-1)^2 - k^2]} \right] \quad (14) \end{aligned}$$

Following the same procedure with the remaining three indices, one obtains:

$$\begin{aligned}
c_2 y_2(r) = a_0' r^2 & \left[ 1 + \frac{(ar)^4}{4 \cdot 6 (25-k^2)} + \frac{(ar)^8}{4 \cdot 6 \cdot 8 \cdot 10 (25-k^2)(81-k^2)} + \right. \\
& + \frac{(ar)^{12}}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 (25-k^2)(81-k^2)(169-k^2)} + \dots + \\
& \left. + \frac{(ar)^{4j}}{2^{2j} \cdot (2j+1)! (25-k^2)(81-k^2) \dots [(4j+1)^2 - k^2]} \right] \quad (15)
\end{aligned}$$

$$\begin{aligned}
c_3 y_3(r) = a_0'' r^{1+k} & \left[ 1 + \frac{(ar)^4}{8(2+k)(3+k)(5+k)} + \right. \\
& + \frac{(ar)^8}{8 \cdot 16(2+k)(3+k)(4+k)(5+k)(7+k)(9+k)} + \\
& + \frac{(ar)^{12}}{8 \cdot 16 \cdot 24(2+k)(3+k)(4+k)(5+k)(6+k)(7+k)(9+k)(11+k)(13+k)} + \dots + \\
& \left. + \frac{(ar)^{4j}}{2^{3j} \cdot j! (2+k)(3+k) \dots (2j+k)(4j+1+k)(4j-1+k)} \right] \quad (16)
\end{aligned}$$

$$\begin{aligned}
C_4 Y_4(r) = a_0''' r^{1-k} & \left[ 1 + \frac{(ar)^4}{8(2-k)(3-k)(5-k)} + \right. \\
& + \frac{(ar)^8}{8 \cdot 16(2-k)(3-k)(4-k)(5-k)(7-k)(9-k)} + \dots + \\
& \left. + \frac{(ar)^{4j}}{2^{3j} \cdot j! (2-k)(3-k) \dots (2j-k)(4j+1-k)(4j-1-k)} \right] \quad (17)
\end{aligned}$$

By using the ratio test, it can easily be shown that the four solutions above converge for all values of  $k$ .

The functions  $Y_1(r)$ ,  $Y_2(r)$ ,  $Y_3(r)$  and  $Y_4(r)$  are solutions corresponding to the case for which  $k$ , the square root of the ratio of two rigidities, is not an integer. For the case  $k=1$ , the four solutions reduce to two.

The solution is then:

$$w(r,t) = [C_1 Y_1(r) + C_2 Y_2(r) + C_3 Y_3(r) + C_4 Y_4(r)] e^{i\omega t} \quad (18)$$

where  $Y_1$ ,  $Y_2$ ,  $Y_3$ , and  $Y_4$  are the functions obtained from Eqs. (14) through (17) and

$$\omega = \alpha^2 \sqrt{\frac{g_r^D}{\gamma h}} \quad (19)$$

$$f = \frac{\omega}{2\pi} = \frac{\alpha^2}{2\pi} \sqrt{\frac{g_r^D}{\gamma h}}$$

The four constants  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  can be determined from the boundary conditions for an annulus. For a solid plate  $Y_4$  is inadmissible because of its singularity at  $r = 0$ . For this case we write  $C_4 = 0$ . Also for symmetrical vibrations it can be shown that  $C_2 = 0$ . The shear in the plate at radius  $r$  must balance the inertia forces as follows:

$$2\pi r Q_r = \int_S \frac{\gamma h}{g} \frac{\partial^2 w}{\partial t^2} dS \quad (20)$$

where  $Q_r$  is the shear and the displacement and acceleration are:

$$w(r,t) = (C_1 Y_1 + C_2 Y_2 + C_3 Y_3) e^{i\omega t}$$

$$\frac{d^2 w}{dt^2} = -\omega^2 (C_1 Y_1 + C_2 Y_2 + C_3 Y_3) e^{i\omega t}$$

The shear is given by:

$$Q_r = - \left[ D_r \left( \frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} \right) - \frac{D_\theta}{r^2} \frac{dw}{dr} \right] e^{i\omega t}$$

and so we have as a condition to be satisfied

$$\frac{r Q_r}{D_r} = - \left[ r \frac{d^3 w}{dr^3} + \frac{d^2 w}{dr^2} - \frac{k^2}{r} \frac{dw}{dr} \right] \quad (21)$$

Substituting the values of the derivatives of the first four terms of  $Y_1$ ,  $Y_2$  and  $Y_3$  into Eq. (21), we obtain finally:

$$- C_2[2(1-k^2)] = 0$$

and  $C_2 = 0$ , since  $k \neq 1$ . Hence, the solution for a solid plate becomes:

$$w(r,t) = [C_1 Y_1 + C_3 Y_3] e^{i\omega t} \quad (22)$$

where the two constants,  $C_1$  and  $C_3$ , will have to be determined from the remaining boundary conditions.

#### 1. Clamped Plate

The boundary conditions for a plate clamped at  $r = a$  are:

$$w = 0 \quad \text{at } r = a$$

$$\frac{dw}{dr} = 0 \quad \text{at } r = a$$

Inserting these conditions into Eq. (22), we obtain:

$$C_1 Y_1(\alpha a) + C_3 Y_3(\alpha a) = 0 \quad (23)$$

$$C_1 Y_1'(\alpha a) + C_3 Y_3'(\alpha a) = 0$$

where the prime indicates the first derivative with respect to  $r$ . Eliminating  $C_1$  and  $C_3$  from Eq. (23) above leads to the frequency equation:

$$Y_1(\alpha a)Y_3'(\alpha a) - Y_3(\alpha a)Y_1'(\alpha a) = 0 \quad (24)$$

The roots of Eq. (24) give the values of  $\alpha$ , which when inserted in Eq. (19) yield the values of the frequencies of the clamped plate.

## 2. Simply Supported Plate

For the simply supported plate the boundary conditions become:

$$w = 0 \quad \text{at } r = a$$

$$M_r = \frac{d^2 w}{dr^2} + \frac{\nu \theta}{r} \frac{dw}{dr} = 0 \quad \text{at } r = a$$

Using the boundary conditions the frequency equation becomes:

$$Y_1(\alpha a)[Y_3''(\alpha a) + \frac{\nu \theta}{a} Y_3'(\alpha a)] - Y_3(\alpha a)[Y_1''(\alpha a) + \frac{\nu \theta}{a} Y_1'(\alpha a)] = 0 \quad (25)$$

Again, the roots of the frequency equation (25) are to be inserted into Eq. (19) to determine the frequencies of the simply supported plate.

The first six terms of the infinite series were used in the expansion of Eq. (25), and the first three frequencies were calculated for several sets of assumed values of the elastic constants. The results are shown in Table I.

Table I

Assumed Elastic Constants and Corresponding Frequencies  
Symmetrical Flexural Vibrations of Simply Supported Circular Plate

Elastic Constants			Roots of Freq. Equ.			Theoretical Frequencies		
k	$\nu_0$	$D_r \times 10^{-4}$	$\alpha_1 a$	$\alpha_2 a$	$\alpha_3 a$	$f_1$	$f_2$	$f_3$
.25	.22	10.70	1.581	-	-	449	-	-
.50	.40	4.75	1.905	-	-	435	-	-
	.30	5.20	1.858	-	-	432	-	-
.75	.70	2.64	2.183	5.315	8.460	426	2523	6392
1.00	.75	1.88	2.349	5.496	8.610	416	2276	5587
	.75	1.60	2.349	5.496	8.610	383	2099	5151
1.25	1.00	1.33	2.544	5.703	8.750	410	2063	4856
	.50	1.50	2.436	5.643	8.736	399	2142	5134
1.50	.75	1.08	2.628	5.835	9.000	394	1944	4624
	.50	1.15	2.578	5.813	8.940	392	1990	4708
1.75	.35	0.95	2.681	5.963	9.120	386	1904	4453



## EXPERIMENTS ON CIRCULARLY STIFFENED CIRCULAR PLATES

The apparatus for determining the frequencies of vibration and the nodal patterns is shown in Figure 1. The plate under study was driven at resonance by an electromagnet mounted beneath it on a solid test stand. A rectangular piece of laminated iron was used in order to develop a driving force. It was cemented to the plate through a flat circular washer in order to create a symmetrical point of dynamic loading.

The frequencies and nodal patterns were determined by means of a capacity type of pickup which can be moved across the surface of the plate. A nodal line is detected by the change in shape of a Lissajous figure on a cathode-ray oscilloscope to which the electrical output from the pickup is fed through a preamplifier. On account of the pull-pull nature of the magnet, the frequency from the oscillator is one half the frequency of the driving force. The output from the oscillator is connected to one set of plates of the cathode-ray oscilloscope, and the magnetic drive is connected across the other set of plates.

Since the frequency ratio is 2 to 1, a figure eight Lissajous is shown on the screen of the scope. Crossing the nodal line with the probe or pickup turns the figure eight Lissajous into a horseshoe-shaped figure on one side of the line, and inverts the figure on the opposite side. This fact

provides a precise means of constructing the nodal pattern.

The experimental frequencies and nodal patterns for the plate under study are shown in Figure 2. The theoretical and experimental frequencies for the first three symmetrical modes of vibration are shown in Table II. The theoretical frequencies were obtained from Table I for the elastic constants determined from previous static tests.

Table II

Comparison of Experimentally Determined  
and Calculated Frequencies for  
Symmetrical Vibration

	$f_1$ c.p.s.	$f_2$ c.p.s.	$f_3$ c.p.s.
Experimental	460	1634	3830
Theoretical	392	1990	4708

While the discrepancy between the theoretical and experimental results is fairly large, it may be noted that the theoretical frequencies were based on experimentally determined elastic compliances for the stiffened plate material and therefore subject to some error. Also, the rotatory moment of inertia is not included in the theoretical solution. Its effect is to reduce the higher frequencies as pointed out

in a paper by Thorkildsen and Hoppmann<sup>6</sup>.

A much better agreement with experimental results is to be expected if the stiffeners are more closely spaced and symmetrically disposed with respect to the center plane of the plate, thus providing a better approximation to the condition of homogeneity assumed in the theory.

#### DISCUSSION AND CONCLUSIONS

The reason for the discrepancy between theory and experiment in the present study can reasonably be attributed to the inappropriateness of the design of the experimental plate which does not sufficiently well meet the requirement of homogeneity on which the theory is based. It is clear that homogeneity will be better approximated if a larger number of appropriately proportioned stiffeners are used. Since the radius of the stiffener decreases for stiffeners closer to the center of the plate, the cross-sections must be likewise reduced. The cross-section of the stiffeners used in the present investigation were reduced in accord with a rational plan but it is considered that future research is necessary to answer this question in a satisfactory manner.

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<sup>6</sup>R. L. Thorkildsen and W. H. Hoppmann II, "Effect of Rotatory Inertia on the Frequencies of Vibration of Stiffened Plates", J. Appl. Mechanics, 298 (1959).

It is also clear that if the stiffeners were formed symmetrically with respect to the middle surface and rotatory inertia taken into account the agreement between the theoretical and experimental results would definitely be improved.

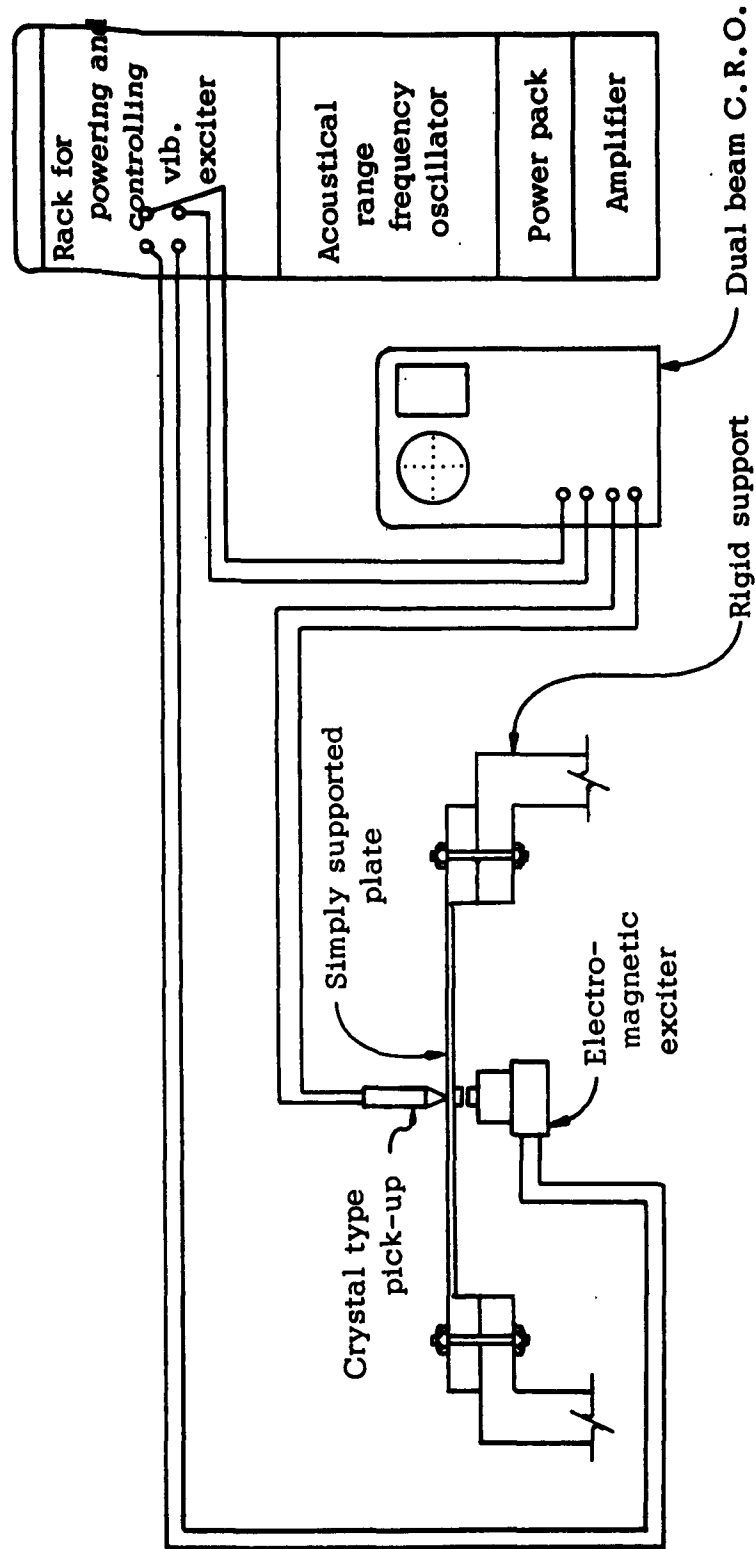
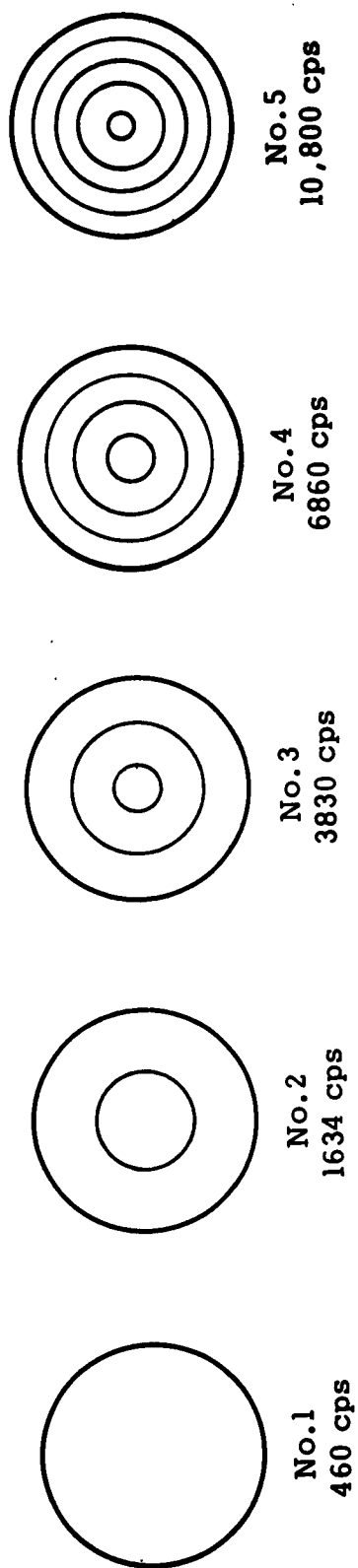
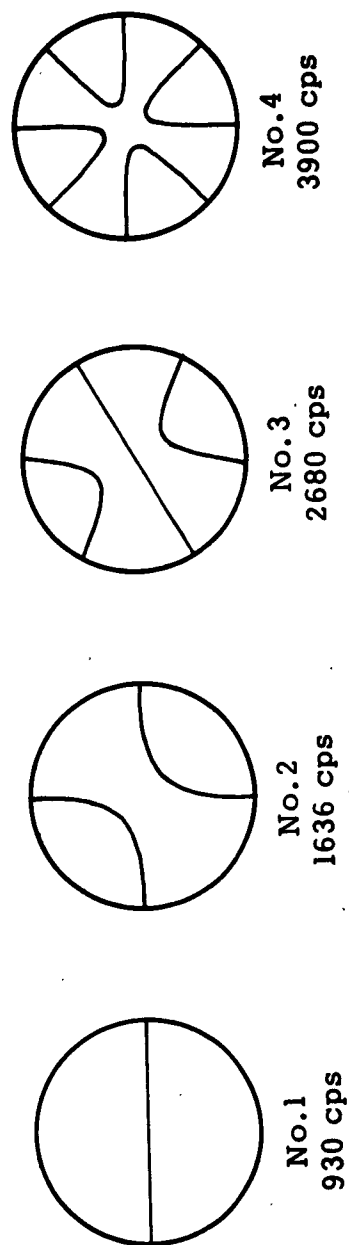


Fig. 1 Experimental Apparatus for Vibrating the Plate



Symmetrical



Asymmetrical

Fig. 2 Nodal Patterns and Corresponding Experimentally Determined Frequencies  
for Simply Supported Stiffened Plate